

The regularity of the positive part of functions in $L^2(I; H^1(\Omega)) \cap H^1(I; H^1(\Omega)^*)$ with applications to parabolic equations

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Abstract. Let $u \in L^2(I; H^1(\Omega))$ with $\partial_t u \in L^2(I; H^1(\Omega)^*)$ be given. Then we show by means of a counter-example that the positive part u^+ of u has less regularity, in particular it holds $\partial_t u^+ \notin L^1(I; H^1(\Omega)^*)$ in general. Nevertheless, u^+ satisfies an integration-by-parts formula, which can be used to prove non-negativity of weak solutions of parabolic equations.

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1 Introduction

In this note, we are concerned with the regularity of the positive part of functions from the function space

$$W := \{u \in L^2(I; H^1(\Omega)) : \partial_t u \in L^2(I; H^1(\Omega)^*)\}$$

of Bochner integrable functions. Here, $I = (0, T)$, $T > 0$, is an open interval, and $H^1(\Omega)$ denotes the usual Sobolev space on the domain $\Omega \subset \mathbb{R}^n$; $\partial_t u$ denotes the weak derivative of u with respect to the time variable $t \in I$. The underlying spaces form a so-called evolution triple (or Gelfand triple) $H^1(\Omega) \subset L^2(\Omega) = L^2(\Omega)^* \subset H^1(\Omega)^*$ with continuous and dense embeddings. In the sequel, we will use the commonly applied abbreviations

$$V := H^1(\Omega), \quad H := L^2(\Omega).$$

For an introduction to these kind of function spaces and their various properties, we refer to e.g. [1, Section IV.1], [3, Section 7.2], [4, Chapter 25].

Let $u \in W$ be given. Let us denote its positive part by u^+ ,

$$u^+(t, x) = \max(u(t, x), 0), \quad t \in I, \quad x \in \Omega.$$

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Due to the embedding $W \hookrightarrow L^2(I \times \Omega)$, the positive part is well-defined. Moreover, since the mapping $u \mapsto u^+$ is bounded from $H^1(\Omega)$ to $H^1(\Omega)$, it follows that for $u \in W$ also $u^+ \in L^2(I; V)$ holds. Here, the question arises whether $u \in W$ also implies $u^+ \in W$. The aim of the short note is to provide an counter-example of this claim, see Theorem 2.7. Nevertheless, the following integration-by-parts formula holds true for all $u \in W$

$$\int_I \langle u_t(s), u^+(s) \rangle_{V^*, V} ds = \frac{1}{2} \|u^+(T)\|_H^2 - \frac{1}{2} \|u^+(0)\|_H^2, \quad (1)$$

which enables us to show positivity of weak solutions of linear parabolic equations, see Section 3.

2 The regularity of the positive part

In this section, we study the mapping properties of $u \mapsto u^+$. First, let us state the following well-known results:

Proposition 2.1. *The mapping $u \mapsto u^+$ is Lipschitz continuous as mapping from H to H . Furthermore it is bounded from V to V , and for $u \in V$ it holds*

$$\nabla u^+(x) = \begin{cases} \nabla u(x) & \text{if } u(x) > 0 \\ 0 & \text{if } u(x) \leq 0 \end{cases}, \quad x \in \Omega,$$

which implies $\|u^+\|_V \leq \|u\|_V$.

The following result is an obvious consequence.

Corollary 2.2. *Let $u \in W$ be given. Then $u^+ \in L^2(I; V) \cap C(\bar{I}; H)$, and it holds*

$$\|u^+\|_{L^2(I; V)}, \|u^+\|_{C(\bar{I}; H)} \leq \|u\|_W.$$

With the same arguments that are classically used to proof Proposition 2.1, one can prove

Corollary 2.3. *Let $u \in W$ be given with $u_t \in L^2(I; H)$. Then $u^+ \in W$ with $u_t^+ \in L^2(I; H)$.*

Moreover, in this case, we have $\partial_t u^+ \in L^2(Q)$, and we can write for almost all $(t, x) \in Q$

$$\partial_t u^+(t, x) = \begin{cases} \partial_t u(t, x) & \text{if } u(t, x) > 0 \\ 0 & \text{if } u(t, x) \leq 0. \end{cases} \quad (2)$$

Now, if $\partial_t u$ is in $L^2(I; V^*)$ only, the representation (2) makes no sense, as $\partial_t u(t, \cdot)$ is only in $H^1(\Omega)^*$ for almost all t .

In the following, we will construct a function $u \in W$ with $\partial_t u \notin L^2(I; H)$ such that $\partial_t u^+ \notin L^2(I; V^*)$. The key idea is the observation that the mapping $u \mapsto u^+$ for $u \in L^2(\Omega)$ is *not* bounded as mapping from $H^1(\Omega)^*$ to $H^1(\Omega)^*$.

To see this, set $\Omega = (0, 1)$. Let us define $\psi_n(x) = \sin(2\pi nx)$. Then it is well-known that ψ_n converges weakly to zero in $L^2(\Omega)$, thus strongly to zero in $H^1(\Omega)^*$. However, a short computation shows that

$$\int_0^1 \psi_n^+(x) dx = \int_0^1 \psi_1^+(x) dx = \int_0^{1/2} \sin(2\pi x) dx = \frac{1}{\pi} \neq 0,$$

which implies that ψ_n^+ converges weakly to the constant function $\hat{\psi}(x) = 1/\pi$ in $L^2(\Omega)$. Hence, ψ_n^+ cannot converge to zero in $H^1(\Omega)^*$.

In the sequel, we will equip V with the scalar product $(u, v)_V := \int_{\Omega} \nabla u \cdot \nabla v + u \cdot v \, dx$ and the associated norm. The space H is equipped with the standard $L^2(\Omega)$ inner product and norm. We consider the family of functions

$$\psi_n(x) := \cos(n\pi x), \quad x \in \Omega \quad (3)$$

for $n \in \mathbb{N}$. Now, we will derive quantitative estimates of the norm of ψ_n in V , H , and V^* for $n \rightarrow \infty$.

Lemma 2.4. *Let $n \in \mathbb{N}$ be given. Then it holds*

$$\|\psi_n\|_V = \left(\frac{n^2\pi^2 + 1}{2} \right)^{1/2} \leq n\pi, \quad \|\psi_n\|_H = \frac{1}{\sqrt{2}}, \quad \|\psi_n\|_{V^*} \leq \frac{1}{\sqrt{2}n\pi}$$

Proof. The first two identities can be verified with elementary calculations. To prove the third, consider the solution $z \in V$ of $(z, v)_V = (\psi_n, v)_H$ for all $v \in V$. Then it follows $\|\psi_n\|_{V^*} = \|z\|_V$. The function z is given by $z = \frac{1}{n^2\pi^2+1}\psi_n$, and hence the third estimate follows from the first. \square

Let us show that the V^* -norm of ψ_n^+ is bounded away from zero.

Lemma 2.5. *There is $C > 0$ such that*

$$\|\psi_n^+\|_{V^*} \geq C \quad \forall n.$$

Proof. Let $e \in H$ be defined by $e(x) = 1$. Then we have

$$\begin{aligned} (\psi_n^+, e)_H &= \int_0^1 \psi_n^+(x) \, dx = \int_0^1 (\cos(n\pi x))^+ \, dx \\ &= n \int_0^{1/2n} \cos(n\pi x) \, dx = \frac{1}{\pi}. \end{aligned}$$

Let now $v_e \in V$ be defined by $v_e(x) = \min(4x, 1, 4(1-x))$. Then it holds $\|v_e - e\|_H^2 = 2 \int_0^{1/4} (4x)^2 \, dx = \frac{1}{6}$. Thus, we can estimate

$$\langle \psi_n^+, v_e \rangle_{V^*, V} \geq (\psi_n^+, e)_H - \|\psi_n^+\|_H \|v - e\|_H \geq \frac{1}{\pi} - \frac{1}{\sqrt{12}} = 0.0296 \dots \geq \frac{1}{5}.$$

Here, we used $\|\psi_n^+\|_H \leq \|\psi_n\|_H = 1/\sqrt{2}$. The lower bound implies that $\|\psi_n^+\|_{V^*} \geq \frac{1}{5}\|v_e\|_V^{-1}$, and the claim is proven. \square

Let us now introduce a family of functions on small time intervals, which will be used to define the counterexample by means of an infinite series.

Lemma 2.6. *Let $I := (0, 1)$. Let $\phi \in H_0^1(I)$ be given. Define*

$$\phi_n(t) := n(n+1) \cdot \phi(n(n+1)t - n). \quad (4)$$

Then it holds $\text{supp } \phi_n \subset \left(\frac{1}{n+1}, \frac{1}{n}\right)$ and

$$\begin{aligned} \|\phi_n\|_{L^1(I)} &= \|\phi\|_{L^1(I)}, & \|\partial_t \phi_n\|_{L^1(I)} &\geq n^2 \|\partial_t \phi\|_{L^1(I)}, \\ \|\phi_n\|_{L^2(I)} &\leq \sqrt{2}n \|\phi\|_{L^2(I)}, & \|\partial_t \phi_n\|_{L^2(I)} &\leq \sqrt{2}n^3 \|\partial_t \phi\|_{L^2(I)}, \end{aligned}$$

Proof. This follows by elementary calculations. \square

Let us now define the function

$$u(x, t) = \sum_{n=1}^{\infty} n^{-3} \phi_n(t) \psi_n(x). \quad (5)$$

Theorem 2.7. *Let $\phi \in H_0^1(I) \setminus \{0\}$ be given with $\phi \geq 0$. Then the function u defined in (5) with ψ_n and ϕ_n from (3) and (4), respectively, belongs to W . However, the time derivative of its positive part $\partial_t u^+$ does not belong to $L^1(I; V^*)$.*

Proof. Let us define the partial sum $u_N := \sum_{n=1}^N \phi_n(t) \psi_n(x)$. We will exploit the fact that the supports of the functions ϕ_n are distinct. From the Lemmas 2.4, 2.5, and 2.6, we have

$$\begin{aligned} \|u_N\|_{L^2(I; V)}^2 &= \sum_{n=1}^N n^{-6} \|\phi_n\|_{L^2(I)}^2 \|\psi_n\|_V^2 \leq c \sum_{n=1}^N n^{-6} \cdot n^2 \cdot n^2 = c \sum_{n=1}^N n^{-2}, \\ \|\partial_t u_N\|_{L^2(I; V^*)}^2 &= \sum_{n=1}^N n^{-6} \|\partial_t \phi_n\|_{L^2(I)}^2 \|\psi_n\|_{V^*}^2 \leq c \sum_{n=1}^N n^{-6} \cdot n^6 \cdot n^{-2} = c \sum_{n=1}^N n^{-2}, \\ \|\partial_t u_N^+\|_{L^1(I; V^*)} &= \sum_{n=1}^N n^{-3} \|\partial_t \phi_n\|_{L^1(I)} \|\psi_n^+\|_{V^*} \geq c \sum_{n=1}^N n^{-3} \cdot n^2 \cdot 1 = c \sum_{n=1}^N n^{-1}. \end{aligned}$$

This proves that (u_N) strongly converges in W to u . Since $u = u_N$ on $\left(\frac{1}{n+1}, 1\right)$, the weak derivative $\partial_t u^+$ exists almost everywhere on I , and belongs to the space $L_{\text{loc}}^1(I; V^*)$. Suppose that $\partial_t u^+ \in L^1(I; V^*)$ holds. Then by the continuity of the integral it follows

$$\|\partial_t u^+\|_{L^1(I; V^*)} = \lim_{N \rightarrow \infty} \int_{1/(N+1)}^1 \|\partial_t u^+(t)\|_{V^*} dt = \lim_{N \rightarrow \infty} \|\partial_t u_N\|_{L^1(I; V^*)} \rightarrow \infty,$$

which is a contradiction, hence $\partial_t u^+ \notin L^1(I; V^*)$. \square

3 Positivity of weak solutions to parabolic equations

Let $\Omega \subset \mathbb{R}^n$ be a domain. Again, we make use of the evolution triple $V = H^1(\Omega)$, $H = L^2(\Omega)$, $V^* = (H^1(\Omega))^*$. Due to the counter-example in the previous section, we cannot apply the well-known integration-by-parts results for functions in W to u^+ . In order to prove formula (1), we recall the following density result

Proposition 3.1. *[3, Lemma 7.2] The space $C^\infty([0, T], V)$ is dense in W .*

First, let us prove the integration-by-parts formula for smooth u .

Lemma 3.2. *Let $u \in W$ with $\partial_t u \in L^2(I; L^2(\Omega))$ be given. Then it holds*

$$\int_0^T \langle \partial_t u(t), u^+(t) \rangle_{V^*, V} dt = \frac{1}{2} \int_0^T \partial_t \|u^+(t)\|_H^2 dt = \frac{1}{2} (\|u^+(t)\|_H^2 - \|u^+(0)\|_H^2). \quad (6)$$

Proof. Since $\partial_t u \in L^2(I; L^2(\Omega))$, it holds $\partial_t u^+ \in L^2(I; L^2(\Omega))$. With the representation (2) it follows

$$\int_Q \partial_t u(x, t) u^+(x, t) \, dx \, dt = \int_Q \partial_t u^+(x, t) u^+(x, t) \, dx \, dt = \frac{1}{2} \int_0^T \partial_t \|u^+(t)\|_H^2 \, dt,$$

which proves the claim. \square

Lemma 3.3. *Let $u \in W$ be given. Then it holds*

$$\int_0^T \langle \partial_t u(t), u^+(t) \rangle_{V^*, V} \, dt = \frac{1}{2} \int_0^T \partial_t \|u^+(t)\|_H^2 \, dt = \frac{1}{2} (\|u^+(t)\|_H^2 - \|u^+(0)\|_H^2).$$

Proof. Let $u \in W$ be given. By density, there is (u_k) in $C^\infty([0, T], V)$ with $u_k \rightarrow u$ in W . By continuity of the projection, it follows $u_k^+ \rightarrow u^+$ in $C([0, T], H)$.

Moreover, the sequence u_k^+ is bounded in $L^2(V)$. Hence, there is a weakly converging subsequence with weak limit \tilde{u} in $L^2(V)$. Due to $u_k^+ \rightarrow u^+$ in $C([0, T], H)$, it follows $\tilde{u} = u^+$, and the whole sequence converges weakly, $u_k^+ \rightharpoonup u^+$ in $L^2(V)$.

Since u_k is smooth enough, u_k satisfies (6). Moreover, the left-hand side and the right-hand side in (6) converge for $k \rightarrow \infty$, proving the claim. \square

Let us remark that this result can be proven using difference quotients, see e.g. [2, Lemma 2.5].

The integration-by-parts formula (1) can be applied to prove non-negativity of weak solutions of parabolic equations with non-negative data. Let $f \in L^1(I; L^2) + L^2(I; V')$ and $u_0 \in H$ be given. Then $u \in W$ is a weak solution of the parabolic equation with homogeneous Neumann boundary conditions

$$\partial_t u - \Delta u = f \text{ on } Q, \quad \partial_n u = 0 \text{ on } I \times \partial\Omega, \quad u(0) = u_0(x), \quad (7)$$

if the following equation is satisfied for all $v \in V$ and almost all $t \in I$

$$\langle \partial_t u(t), v \rangle_{V^*, V} + \int_\Omega \nabla u(x, t) \nabla v(x) \, dx = \langle f(t), v \rangle_{V^*, V}.$$

Theorem 3.4. *Let $f \in L^1(I; L^2(\Omega)) + L^2(I; V^*)$ be given, with $f \geq 0$, which is $\langle f, v \rangle \geq 0$ for all $v \in L^2(V) \cap C(I; H)$ with $v \geq 0$. Let $u_0 \in H$ be given with $u_0 \geq 0$. Let u be a weak solution of the parabolic equation (7). Then it holds $u \geq 0$.*

Proof. Let us denote $u^- = -(-u)^+ \in L^2(V) \cap C(I; H)$. Testing the weak formulation with u^- , integrating from 0 to t , and using Proposition 2.1 and Lemma 3.3 yields

$$\begin{aligned} 0 &\geq \int_0^t \langle f(s), u^-(s) \rangle_{V^*, V} \, ds \\ &= \int_0^t \langle \partial_t u(s), u^-(s) \rangle_{V^*, V} \, ds + \int_0^t \int_\Omega \nabla u(x, s) \nabla u^-(x, s) \, dx \, ds \\ &= \frac{1}{2} (\|u^-(t)\|_H^2 - \|u^-(0)\|_H^2) + \|\nabla u^-\|_{L^2(0, t; L^2(\Omega))}^2 \\ &\geq \frac{1}{2} \|u^-(t)\|_H^2. \end{aligned}$$

Hence, it follows $u^-(t) = 0$ for almost all $t \in I$, which implies $u^- = 0$ almost everywhere on Q . \square

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